The response of interferometers to scalar gravitational waves in the "Shibata, Nakao and Nakamura" gauge: a correct analysis

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Abstract

While the response of interferometers to tensorial gravitational waves has been computed in lots of works, the coupling between interferometers and scalar gravitational waves (SGWs) is a more recent field of interest and it has not an analogous number of works in literature.

In the first work of [1] the authors did not realize that in their gauge the beam splitter is not left at the origin by the passage of the SGW, and furthermore computed a coordinate-time interval than a proper-time interval, reaching the incorrect conclusion that the SGW has longitudinal effect, and does not have transverse one. In [2] the transverse effect of SGWs in the "Shibata Nakao and Nakamura" (SNN) gauge of [1] was shown. The analysis of [2] was generalized in [3] with the computation of the frequency-dependent angular pattern of interferometers in the SNN gauge, while in [2] the angular pattern was only computed in the low frequencies approximation (wavelength much larger than the linear dimensions of the interferometer).

In this paper the SNN gauge is reanalyzed, showing that in [1] there was an error in the geodesic equations of motion too. This error conditioned also the analysis of [2] and [3] where wrong equation of motion taken from [1] were used.

In the analysis of the response of interferometers the computation is first made in the low frequencies approximation, then the analysis is applied to all SGWs using a generalization to the SNN gauge of the analysis of [4], where the computation was made in the TT gauge for tensorial waves.

At the end of this paper the correct detector pattern of interferometers in the SNN gauge is also computed with a further generalization of the analysis of [4] to the angular dependence of the propagating SGW.

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1 Introduction

The design and construction of a number of sensitive detectors for gravitational waves (GWs) is underway today. There are some laser interferometers like the VIRGO detector, being built in Cascina, near Pisa, by a joint Italian-French collaboration, the GEO 600 detector, being built in Hanover, Germany, by a joint Anglo-Germany collaboration, the two LIGO detectors, being built in the United States (one in Hanford, Washington, and the other in Livingston, Louisiana), by a joint Caltech-Mit collaboration, and the TAMA 300 detector, being built near Tokyo, Japan. Many bar detectors are currently in operation too, and several interferometers and bars are in a phase of planning and proposal stages (for the current status of gravitational waves experiments see [5, 6]).

The results of these detectors will have a fundamental impact on astrophysics and gravitation physics. There will be lots of experimental data to be analyzed, and theorists will be forced to interact with lots of experiments and data analysts to extract the physics from the data stream.

Detectors for GWs will also be important to verify that GWs only change distances perpendicular to their direction of propagation and to confirm or ruling out the physical consistency of General Relativity or of any other theory of gravitation [7, 8].

These detectors are in principle sensitive also to a hypotetical *scalar* component of gravitational radiation, that appears in extended theories of gravity like scalar-tensor gravity [7, 8], Brans-Dicke theory [9] and string theory [10, 11].

While the response of interferometers to tensorial waves has been calculated in lots of works (see for example [4, 12, 13]), the coupling between interferometers and scalar waves is a more recent field of interest and it has not an analogous number of works in literature. Here it has to be recalled in particular the first work of [1] which was improved from the work of the authors of [2]. In [1] the authors did not realize that in their gauge the beam splitter is not left at the origin by the passage of the SGW, and furthermore computed a coordinate-time interval than a proper-time interval, reaching the incorrect conclusion that the SGW has longitudinal effect, and does not have transverse one. In [2] the transverse effect of SGWs was shown in the SNN gauge of [1]. After this, in [3], the analysis of [2] was generalized with the computation of the frequency-dependent angular pattern of interferometers in the SNN gauge, while in [2] the angular pattern was only computed in the low frequencies approximation (wavelength much larger than the linear dimensions of the interferometer, under this assumption the amplitude of the SGW,Φ, can be considered "frozen" at a value Φ_0).

In this paper the SNN gauge is reanalyzed, showing that in [1] there were errors in the geodesic equations of motion too. These errors reflected also in [2] and [3] where wrong equation of motion taken from [1] were used.

In the analysis of the response of interferometers the computation is first made in the low frequencies approximation exactly like in [2], then the calculation is generalized to all SGWs. In the generalization to all frequencies, with a transform of the time coordinate to the proper time, a generalization to the SNN gauge of the analysis of [4] will be used (in [4] the analysis was made in the TT gauge for tensorial waves).

At the end of this paper, the correct detector pattern of interferometers in the SNN gauge is computed with a further generalization of the analysis of [4] to the angular dependence of the propagating SGW.

2 The "Shibata Nakao and Nakamura" gauge for scalar gravitational waves

It will be considered a gauge which was proposed in the first time in ref. [1]. In this gauge a purely plane scalar gravitational wave is travelling in the $z+$ direction (progressive wave $\Phi \equiv \Phi(t-z)$) and acting on an interferometer whose arms are aligned along the x and z axes [1, 2]). In this gauge it is [1, 2, 3]

$$e_{\mu\nu}^{(s)} = \eta_{\mu\nu}, \tag{1}$$

thus the line element results (we work with $c = 1$ and $\hbar = 1$ in this paper)

$$ds^2 = [1 + \Phi(t-z)](-dt^2 + dz^2 + dx^2 + dy^2). \tag{2}$$

Eq. (1) can be rewrite as

$$(\frac{dt}{d\tau})^2 - (\frac{dx}{d\tau})^2 - (\frac{dy}{d\tau})^2 - (\frac{dz}{d\tau})^2 = \frac{1}{(1+\Phi)}, \tag{3}$$

where τ is the proper time of the test masses.

From eqs. (2) and (3) the authors of [1] obtained the geodesic equations of motion for test masses (i.e. the beam-splitter and the mirrors of the interferometer), see eq. 3.21, 3.22 and 3.23 of [1],

$$\begin{array}{ccc}
\frac{d}{d\tau}[(1+\Phi)\frac{dx}{d\tau}] & = & 0 \\
\\
\frac{d}{d\tau}[(1+\Phi)\frac{dy}{d\tau}] & = & 0 \\
\\
\frac{d}{d\tau}[(1+\Phi)\frac{dt}{d\tau}] & = & \frac{1}{2}\frac{\partial_t(1+\Phi)}{(1+\Phi)} \\
\\
\frac{d}{d\tau}[(1+\Phi)\frac{dz}{d\tau}] & = & -\frac{1}{2}\frac{\partial_z(1+\Phi)}{(1+\Phi)},
\end{array} \tag{4}$$

which are wrong!

Other wrong geodesic equations of motion are used in [3], see eqs. 4.2, 4.3, 4.4 and 4.5, in this case for a wave travelling in the $z-$ direction (regressive wave), thus the results of [3] and in particular the frequency-dependent angular pattern of eq. (5.25) are not correct.

To derive the correct geodesic equation of motion for a progressive wave, eq. (87,3) of ref. [14], which is

$$\frac{d^2x^i}{d\tau^2} + \Gamma^i_{kl}\frac{dx^k}{d\tau}\frac{dx^l}{d\tau} = 0, \tag{5}$$

can be used.

In this way, using equation (3), it results

$$\begin{array}{ccc} \frac{d^2x}{d\tau^2} & = & 0 \\ \\ \frac{d^2y}{d\tau^2} & = & 0 \\ \\ \frac{d^2t}{d\tau^2} & = & \frac{1}{2}\frac{\partial_t(1+\Phi)}{(1+\Phi)^2} \\ \\ \frac{d^2z}{d\tau^2} & = & -\frac{1}{2}\frac{\partial_z(1+\Phi)}{(1+\Phi)^2}. \end{array} \tag{6}$$

But which is the difference between the correct eqs. (6) and the wrong ones (4)? To understand this let us rewrite eqs. (4) as

$$\begin{array}{ccc} d[(1+\Phi)\frac{dx}{d\tau}] & = & 0 \\ \\ d[(1+\Phi)\frac{dy}{d\tau}] & = & 0 \\ \\ d[(1+\Phi)\frac{dt}{d\tau}] & = & \frac{1}{2}\frac{\partial_t(1+\Phi)}{(1+\Phi)}d\tau \\ \\ d[(1+\Phi)\frac{dz}{d\tau}] & = & -\frac{1}{2}\frac{\partial_z(1+\Phi)}{(1+\Phi)}d\tau, \end{array} \tag{7}$$

It will be shown that the total differential of every component of the metric tensor is equal to zero (i.e. the metric tensor works like a constant in the total differential). In fact it is $du_i = g_{ik}du^k$, but it is also $du_i = d(g_{ik}u^k) = (dg_{ik})u^k + g_{ik}du^k$, thus it has to be

$$dg_{ik} = 0. \tag{8}$$

In this way eqs. (7) can be rewritten as

$$\begin{array}{ccc} d[(1+\Phi)\frac{dx}{d\tau}] & = (1+\Phi)d[\frac{dx}{d\tau}] = & 0 \\ \\ d[(1+\Phi)\frac{dy}{d\tau}] & = (1+\Phi)d[\frac{dy}{d\tau}] = & 0 \\ \\ d[(1+\Phi)\frac{dt}{d\tau}] & = (1+\Phi)d[\frac{dt}{d\tau}] = & \frac{1}{2}\frac{\partial_t(1+\Phi)}{(1+\Phi)}d\tau \\ \\ d[(1+\Phi)\frac{dz}{d\tau}] & = (1+\Phi)d[\frac{dz}{d\tau}] = & -\frac{1}{2}\frac{\partial_z(1+\Phi)}{(1+\Phi)}d\tau, \end{array} \tag{9}$$

that becomes

$$\begin{array}{ccc} \frac{d^2x}{d\tau^2} & = & 0 \\ \frac{d^2y}{d\tau^2} & = & 0 \\ \frac{d^2t}{d\tau^2} & = & \frac{1}{2}\frac{\partial_t(1+\Phi)}{(1+\Phi)^2} \\ \frac{d^2z}{d\tau^2} & = & -\frac{1}{2}\frac{\partial_z(1+\Phi)}{(1+\Phi)^2}, \end{array} \tag{10}$$

which are exactly eqs. (6).

The fact that the metric tensor works like a constant in the total derivative has to be emphasized if one uses eq. (87,3a) of ref. [14], which is

$$\frac{du_i}{d\tau}-\frac{1}{2}\frac{\partial g_{kl}}{\partial x^i}u^k u^l=0, \tag{11}$$

in the derivaion of the geodesic equations of motion. In fact, in this case, if one does not recall that the metric tensor works like a constant in the total derivative, eqs. (4) are directly obteined from eq. (11)!

In [1] the authors did not realize that the metric tensor works like a constant in the total derivative and this will be fundamental for the conservation of some quantities in the next analysis.

The first and the second of eqs. (6) can be immediately integrated obtaining

$$\frac{dx}{d\tau}=C_1=const. \tag{12}$$

$$\frac{dy}{d\tau}=C_2=const. \tag{13}$$

In this way eq. (3) becomes

$$(\frac{dt}{d\tau})^2-(\frac{dz}{d\tau})^2=\frac{1}{(1+\Phi)}. \tag{14}$$

Assuming that test masses are at rest initially it results $C_1=C_2=0$. Thus, even if the SGW arrives at test masses, there is not motion of test masses within the $x-y$ plane in this gauge. This fact could be directly understood from eq. (2) because the absence of the x and of the y dependences in the metric implies that test masses momentum in these directions (i.e. C_1 and C_2 respectively) is conserved. This results, for example, from the fact that in this case the x and y coordinates do not esplicitly enter in the Hamilton-Jacobi equation for a test mass in a gravitational field (see ref. [14]).

Now it will be shown that, in presence of a SGW, there is motion of test masses in the z direction which is the direction of the propagating wave. An analysis of eqs. (6) shows that, to simplify equations, the retarded and advanced time coordinates (u,v) can be introduced, exactly like in [1]:

$$u = t - z$$
$$v = t + z. \tag{15}$$

From the third and the fourth of eqs. (6) it is

$$\frac{d}{d\tau}\frac{du}{d\tau} = \frac{\partial_v[1+\Phi(u)]}{(1+\Phi(u))^2} = 0. \tag{16}$$

Equation (16) represents the fundamental difference with the work of [1]: the authors of [1] found the equation (see eq. (3.27) of [1])

$$\frac{d}{d\tau}([1+\Phi(u)]\frac{du}{d\tau}) = \frac{\partial_v[1+\Phi(u)]}{1+\Phi(u)} = 0, \tag{17}$$

which was integrated obtaining (eq. (3.28) of [1])

$$\frac{du}{d\tau} = \frac{a}{1+\Phi}, \tag{18}$$

while, using eq. (16) we obtain

$$\frac{du}{d\tau} = \alpha, \tag{19}$$

where α is an integration constant.
From eqs. (14) and (19), it is also

$$\frac{dv}{d\tau} = \frac{\beta}{1+\Phi} \tag{20}$$

where $\beta \equiv \frac{1}{\alpha}$, and

$$\tau = \beta u + \gamma, \tag{21}$$

where the integration constant γ correspondes simply to the retarded time coordinate translation u. Thus, without loss of generality, it can be put equal to zero.

Instead in [1] the authors found (eq. (3.29) of [1])

$$\frac{dv}{d\tau} = \frac{1}{a} \tag{22}$$

and (eq. (3.30) of [1])

$$\tau = av + b. \tag{23}$$

The difference between eqs. (10) and eqs. (7) generates the differences between the wrong eqs. (3.28) and (3.29) in [1] and the correct eqs. (19) and (20) of this paper, i.e. in our work we obtain the conservation of $\frac{du}{d\tau}$ while the authors of [1] have found the conservation of $\frac{dv}{d\tau}$.

Now let us see what is the meaning of the other integration constant β (see also [1]). From eqs. (19) and (20) the equation for z can be written:

$$\frac{dz}{d\tau} = \frac{1}{2\beta}(\frac{\beta^2}{1+\Phi} - 1). \tag{24}$$

When it is $\Phi = 0$ (i.e. before the SGW arrives at the test masses) eq. (24) becomes

$$\frac{dz}{d\tau} = \frac{1}{2\beta}(\beta^2 - 1). \tag{25}$$

But this is exactly the initial velocity of the test mass, thus $\beta = 1$ has to be choosen because test masses are supposed at rest initially. This also imply $\alpha = 1$.

To find the motion of a test mass in the z direction, we note that from eq. (21) it is $d\tau = du$, while from eq. (20) it is $dv = \frac{d\tau}{1+\Phi}$.

Because it is also $z = \frac{v-u}{2}$ we obtain

$$dz = \frac{1}{2}(\frac{d\tau}{1+\Phi} - du), \tag{26}$$

which can be integred as

$$\begin{array}{c} z = z_0 + \frac{1}{2}\int(\frac{du}{1+\Phi} - du) = \\ \\ = z_0 - \frac{1}{2}\int_{-\infty}^{t-z}\frac{\Phi(u)}{1+\Phi(u)}du, \end{array} \tag{27}$$

where z_0 is the initial position of the test mass. Now the displacement of the test mass in the z direction can be written as

$$\begin{array}{c} \Delta z = z - z_0 = -\frac{1}{2}\int_{-\infty}^{t-z_0-\Delta z}\frac{\Phi(u)}{1+\Phi(u)}du \\ \\ \simeq -\frac{1}{2}\int_{-\infty}^{t-z_0}\frac{\Phi(u)}{1+\Phi(u)}du. \end{array} \tag{28}$$

Our results can be also rewritten in function of the time coordinate t:

$$\begin{array}{ccc} x(t) & = & x_0 \\ \\ y(t) & = & y_0 \\ \\ z(t) & = & z_0 - \frac{1}{2}\int_{-\infty}^{t-z_0}\frac{\Phi(u)}{1+\Phi(u)}d(u) \\ \\ \tau(t) & = & t - z(t), \end{array} \tag{29}$$

which are different from

$$\begin{array}{rcl} x(t) & = & x_0 \\ y(t) & = & y_0 \\ z(t) & = & z_0+\frac{1}{2}I(t-z(t)) \\ \tau(t) & = & t+z(t) \end{array} \tag{30}$$

with

$$I(t-z(t)) \equiv \int_{-\infty}^{t-z_0} \Phi(u)du \tag{31}$$

used from the authors of [2] starting from the wrong geodesic equations (4). In [3], for a regressive wave (i.e. in this case it is $\Phi \equiv \Phi(t+z)$), it is also:

$$\begin{array}{rcl} \overline{x} & = & \overline{x}_i \\ \overline{y} & = & \overline{y}_i \\ \overline{z} & = & \overline{z}_i - \frac{1}{2}\int_{-\infty}^{t+\overline{z}_i} \delta\Phi(v)d(v) \\ \frac{d\tau}{dv} & = & 1+\delta\Phi, \end{array} \tag{32}$$

see eqs. 4.11 - 4.14, which are wrong too (i.e. in [3] the scalar field is indicated with $\delta\Phi$, and the coordinates are barred: in this paper we use the same notations of [3] only in eq. (32) and (33)). With an analysis analogous to the one used above, it is simple to show that the correct equations of motion for a regressive SGW are

$$\begin{array}{rcl} \overline{x} & = & \overline{x}_i \\ \overline{y} & = & \overline{y}_i \\ \overline{z} & = & \overline{z}_i + \frac{1}{2}\int_{-\infty}^{t+\overline{z}_i} \frac{\delta\Phi(v)}{1+\delta\Phi(v)}d(v) \\ \tau & = & t+z, \end{array} \tag{33}$$

Now we reasume what happens in the SNN gauge: it has been shown that in the $x-y$ plane an inertial test mass initially at rest remains at rest throughout the entire passage of the SGW, while in the z direction an inertial test mass initially at rest has a motion during the passage of the SGW. Thus it could appear that SGWs have a longitudinal effect and do not have a transversal one (incorrect conclusion of [1]), but the situation is different as it will be shown in the following analysis.

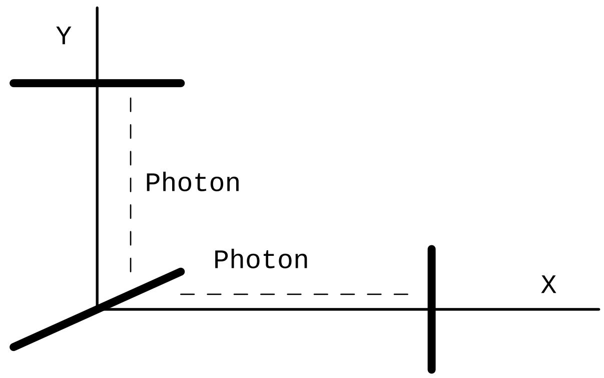

Figure 1: photons can be launched from the beam-splitter to be bounced back by the mirror

3 Analysis in the low frequencies approximation

We have to clarify the use of words " at rest" : we want to mean that the coordinates of test masses do not change in the presence of the SGW in the $x-y$ plane, but it will be shown that the proper distance between the beam-splitter and the mirror of our interferometer changes even though their coordinates remain the same. On the other hand, it will be also shown that the proper distance between the beam-splitter and the mirror of our interferometer does not change in the z direction even if their coordinates change in the SNN gauge (2).

A good way to analyze variations in the proper distance (time) is by means of "bouncing photons" : a photon can be launched from the beam-splitter to be bounced back by the mirror (see ref. [4] and figure 1).

In this section we only deal with the case in which the frequency f of the SGW is much smaller than $\frac{1}{T_0} = \frac{1}{L_0}$, where $2T_0 = 2L_0$ is the total round-trip time of the photon in absence of the SGW, exactly like in [2], but the correct eqs. (29) will be used differently from the authors of [2] that used the wrong ones (30). The analysis will be generalized to all frequencies in the next section.

Assuming that test masses are located along the x axis and the z axis of the coordinate system the y direction can be neglected because the absence of the y dependence in the metric (2) implies that photon momentum in this direction is conserved (refs. [4, 14]), and the interval can be rewritten in the form

$$ds^2 = [1 + \Phi(t - z)](-dt^2 + dx^2 + dz^2). \tag{34}$$

Let us start by considering the interval for a photon which propagates in the x axis. Photon momentum in the z direction is not conserved, for the z dependence in eq. (2) (refs. [4, 14]). Thus photons launched in the x axis will deflect out of this axis. But this effect can be neglected because the photon deflection into the z direction will be at most of order Φ [4]. Then, to first order in Φ, the dz^2 term can be neglected. Thus, from eq. (34) it results

$$ds^2 = (1+\Phi)(-dt^2) + (1+\Phi)dx^2. \tag{35}$$

The condition for null geodesics ($ds^2 = 0$) for photons gives

$$\frac{dx_{photon}}{dt} = \pm 1 \Rightarrow x_{photon} = const \pm t. \tag{36}$$

In the SNN gauge, the x coordinates of the beam-splitter and the mirrors are unaffected by the passage of the SGW (see the first of eqs. (29)), then, from eq. (36) it results that the interval, in coordinate time t, that the photon takes for run one round trip in the x arm of the interferometer is

$$T = 2L_0 \tag{37}$$

(i.e. the photon leaves the beam-splitter at $t = 0$ and returns a $t = T$). But this quantity is not invariant under coordinate transformations [2], and we have to work in terms of the beam-splitter proper time which misures the physical lenght of the arms. In this way we call $\tau(t)$ and $z_b(t)$ the proper time and z coordinate of the beam-splitter at time coordinate t with initial condition $z_b(-\infty) = 0$. From eqs. (29) it is

$$\begin{array}{lll} z_b(t) & = & -\frac{1}{2}\int_{-\infty}^{t-z_b(t)} \frac{\Phi(u)}{1+\Phi(u)}du \\ \\ \tau(t) & = t & +\frac{1}{2}\int_{-\infty}^{t-z_b(t)} \frac{\Phi(u)}{1+\Phi(u)}du. \end{array} \tag{38}$$

Thus, calling τ_x the proper time interval that the photon takes to run a round-trip in the x arm, it is

$$\begin{array}{c} \tau_x = \tau(T) - \tau(0) = T + \frac{1}{2}\int_{-z_b(0)}^{t-z_b(t)} \frac{\Phi(u)}{1+\Phi(u)}du \simeq \\ \\ \simeq T + \frac{1}{2}\Phi_0[T + z_b(0) - z_b(T)] \simeq \\ \\ \simeq 2L_0(1+\frac{1}{2}\Phi_0), \end{array} \tag{39}$$

which is the same result obteined in [2], but now it is obteined from the correct equations of motion.

In the above computation eq. (37) have been used and, by considering only the first order in Φ with $\Phi \ll 1$, the scalar field Φ has also been considered "frozen" at a fixed value Φ_0. Note that $z_b(0) - z_b(T)$ is second order in Φ_0.

The computation of [2] is correct but it starts from wrong equations of motion, i.e. the authors of [2] casually obteined the correct result (39) starting from wrong equations of motion. This is because the correct equation of motion (33) for a regressive SGW are casually very similiar to the wrong ones (30) for a progressive SGW. In fact, rewritting the correct equations of motion for a regressive wave (33) and using the notation of [2] it results

$$
\begin{aligned}
x(t) &= x_0 \\
y(t) &= y_0 \\
z(t) &= z_0 + \frac{1}{2}\int_{-\infty}^{t+z_0} \frac{\Phi(u)}{1+\Phi(u)} d(u) \\
\tau(t) &= t + z(t),
\end{aligned}
\tag{40}
$$

and, because it is $\Phi(u) \ll 1$ we have

$$
\frac{\Phi(u)}{1+\Phi(u)} \simeq \Phi(u), \tag{41}
$$

and the only difference between eqs. (40) and eqs. (30) is the different parametrization of the wave: regressive in eq. (40), progressive in eq. (30).

Now let us consider the z direction: the x direction can be neglected because the absence of the x dependence in the metric (34) implies that photon momentum in this direction is conserved (refs. [14]). From eq. (34) it is now:

$$
ds^2 = (1+\Phi)(-dt^2) + (1+\Phi)dz^2, \tag{42}
$$

and the condition for null geodesics ($ds^2 = 0$) for photons gives

$$
\frac{dz_{photon}}{dt} = \pm 1 \Rightarrow z_{photon} = const \pm t. \tag{43}
$$

We suppose that the photon leaves the beam splitter at $t = 0$; let us ask: how much time does the photon need to arrive at the mirror in the z axis? Calling T_1 this time we need the condition

$$
z_b(0) + T_1 = z_m(T_1), \tag{44}
$$

where $z_m(t)$ is the z coordinate of the mirror in the z axis at coordinate time t with $z_m(-\infty) = L_0$. In the same way, when returning from the mirror, the photon arrives again at the beam-splitter at $t = T_z = T_1 + T_2$, then

$$
z_m(T_1) - T_2 = z_b(T_z). \tag{45}
$$

Subtracting eq. (45) from eq. (44) it is

$$
T_z = T_1 + T_2 = [z_m(T_1) - z_b(0)] + [z_m(T_1) - z_b(T_z)]. \tag{46}
$$

From eq. (29) the equations of motion for z_b and z_m are:

$$
\begin{aligned}
z_m(t) &= L_0 - \frac{1}{2}\int_{-\infty}^{t-z_m(t)} \frac{\Phi(u)}{1+\Phi(u)} du \\
z_b(t) &= -\frac{1}{2}\int_{-\infty}^{t-z_b(t)} \frac{\Phi(u)}{1+\Phi(u)} du,
\end{aligned}
\tag{47}
$$

and, substituing them in eq. (46), it results

$$T_z = 2L_0 - \frac{1}{2}\int_{-z_b(0)}^{T_1 - z_m(T_1)} \frac{\Phi(u)}{1+\Phi(u)}du - \frac{1}{2}\int_{T_z - z_b(T_z)}^{T_1 - z_m(T_1)} \frac{\Phi(u)}{1+\Phi(u)}du. \tag{48}$$

From eq. (44) the first integral in eq. (48) is zero. The second integral is simple to compute considering the SGW frozen at a value Φ_0. To first order in this value it is

$$\begin{array}{c} -\frac{1}{2}\int_{T_z - z_b(T_z)}^{T_1 - z_m(T_1)} \frac{\Phi(u)}{1+\Phi(u)}du \simeq -\frac{1}{2}\Phi_0[T_1 - z_m(T_1) - T_z + z_b(T_z)] \simeq \\ \simeq -\frac{1}{2}\Phi_0(L_0 - L_0 - 2L_0) = +\frac{1}{2}\Phi_0 2L_0. \end{array} \tag{49}$$

In this way eq. (48) becomes

$$T_z = (1 + \frac{1}{2}\Phi_0)2L_0. \tag{50}$$

Then, calling τ_z the proper time interval that the photon takes to run a round-trip in the z arm, with the same way of thinking which leaded to eq. (39), it results

$$\begin{array}{c} \tau_z = \tau(T_z) - \tau(0) = T_z + \frac{1}{2}\int_{-z_b(0)}^{t - z_b(t)} \frac{\Phi(u)}{1+\Phi(u)}du \simeq \\ \simeq T_z + \frac{1}{2}\Phi_0[T_z + z_b(0) - z_b(T)] \simeq \\ \simeq T_z(1 - \frac{1}{2}\Phi_0) \simeq 2L_0, \end{array} \tag{51}$$

which is also the same result of the correspondent equation in [2]:

$$\tau_z \simeq T_z(1 + \frac{1}{2}\Phi_0) \simeq 2L_0. \tag{52}$$

This is also due to the similiarity of eqs. (40) and eqs. (30): the difference in sign between eqs. (51) and (52) is due to the difference between the progressive and the regressive wave. In [2] it is also

$$T_z = (1 - \frac{1}{2}\Phi_0)2L_0, \tag{53}$$

i.e. a difference in sign with eq. (50), which compensates the difference in sign between eqs. (51) and (52).

Thus, from eqs. (39) and (51) it is possible to say that there is a variation of the proper distance in the x direction (transverse effect of the SGW), while there is not a variation of the proper distance in the z direction (no longitudinal effect).

4 Generalized analysis

Now we generalize to all the frequencies the previous result, with an analysis that, with a transform of the time coordinate to the proper time, generalizes to the SNN gauge the analysis of [4], where the analysis was made in the TT gauge for tensorial waves. In this way the response function of the interferometer to SGWs in the SNN gauge will be obtained.

Let us start with the x arm of the interferometer. The condition of null geodesic (36) can be also rewrite in this way:

$$dt^2 = dx^2. \tag{54}$$

In ref. [4] the author used the condition of null geodesic in the case of the TT gauge for tensorial waves to obtain the coordinate velocity of the photon which was used for calculations of the photon propagation times between the test masses (eq. (4) in [4]). But from eq. (54) it results that the coordinate velocity of the photon in the SNN gauge is equal to the speed of light. Thus, in this case, the analysis of [4] cannot be used starting directly from the condition of null geodesic. Then let us ask which is the important difference between the SNN gauge and the TT gauge for tensorial waves analyzed in [4]. The answer is that the TT gauge of [4] is a "synchrony gauge", a coordinate system in which the time coordinate t is exactly the proper time (about the synchrony coordinate system see Cap. (9) of ref. [14]). In the coordinates (2) t is only a time coordinate. The rate $d\tau$ of the proper time is related to the rate dt of the time coordinate from [14]

$$d\tau^2 = g_{00}dt^2. \tag{55}$$

Only making the time transform (55) the analysis of [4] can be applied to the SNN gauge.

From eq. (35) it is $g_{00} = (1 + \Phi)$. Then, using eq. (54), we obtain

$$d\tau^2 = (1 + \Phi)dx^2, \tag{56}$$

which gives

$$d\tau = \pm(1 + \Phi)^{\frac{1}{2}}dx. \tag{57}$$

Now it will be shown that the analysis of [4] works in this case too.

From eqs. (29) the coordinates of the beam-splitter $x_b = l$ and of the mirror $x_m = l + L_0$ do not changes under the influence of the SGW in the SNN gauge, thus the proper duration of the forward trip can be found as

$$\tau_1(t) = \int_l^{L_0+l} [1 + \Phi(t)]^{\frac{1}{2}}dx. \tag{58}$$

To first order in Φ this integral can be approximated with

$$\tau_1(t) = T_0 + \frac{1}{2}\int_l^{L_0+l} \Phi(t')dx \tag{59}$$

where

$$t' = t - (l + L_0 - x).$$

In the last equation t' is the retardation time (i.e. t is the time at which the photon arrives in the position $l + L_0$, so $l + L_0 - x = t - t'$ [4]).

In the same way, for the proper duration of the return trip, it is

$$\tau_2(t) = T_0 + \frac{1}{2}\int_{l+L_0}^{l} \Phi(t')(-dx) \tag{60}$$

where now

$$t' = t - (x - l)$$

is the retardation time and

$$T_0 = L_0$$

is the transit proper time of the photon in the absence of the SGW, which also corresponds to the transit coordinate time of the photon in the presence of the SGW (see eq. (54)).

Thus the round-trip proper time will be the sum of $\tau_2(t)$ and $\tau_1(t - T_0)$. Then, to first order in Φ, the proper duration of the round-trip will be

$$\tau_{r.t.}(t) = \tau_1(t - T_0) + \tau_2(t). \tag{61}$$

By using eqs. (59) and (60) it appears immediately that deviations of this round-trip proper time (i.e. proper distance) from its imperurbated value are given by

$$\delta\tau(t) = \frac{1}{2}\int_l^{L_0+l} [\Phi(t - 2T_0 + x - l) + \Phi(t - x + l)]dx. \tag{62}$$

Eq. (62) generalizes eq. (39) which was derived in the low frequencies approximation. We can also define the signal seen from the arm in the x axis like

$$\frac{\delta\tau(t)}{T_0} \equiv \frac{1}{2T_0}\int_l^{L_0+l} [\Phi(t - 2T_0 + x - l) + \Phi(t - x + l)]dx. \tag{63}$$

Now the analysis will be transled in the frequency domain using the Fourier transform of the field defined by

$$\tilde{\Phi}(\omega) = \int_{-\infty}^{\infty} dt\Phi(t)\exp(i\omega t). \tag{64}$$

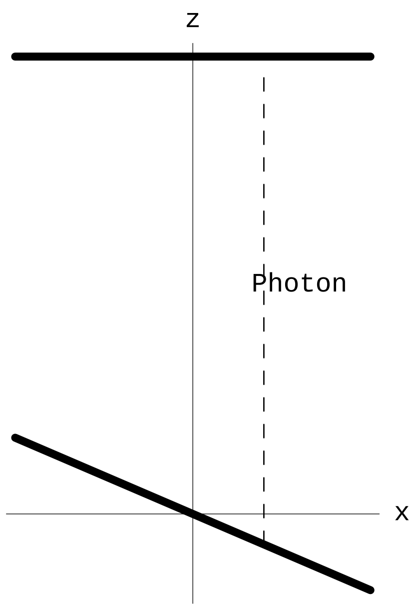

Figure 2: the beam splitter and the mirror are located in the direction of the incoming GW

By using definition (64), from eq. (63) it is

$$\frac{\delta\tilde{\tau}(\omega)}{T_0} = \Upsilon(\omega)\tilde{\Phi}(\omega), \tag{65}$$

where $\Upsilon(\omega)$ is the response of the x arm of our interferometer to SGWs:

$$\Upsilon(\omega) = \frac{\exp(2i\omega T_0) - 1}{2i\omega T_0}. \tag{66}$$

Now let us see what happens in the z coordinate (see figure 2).
From eq. (42) and the condition $ds^2 = 0$ for null geodesics it also results

$$dz = \pm dt. \tag{67}$$

But, from the last of eqs. (29) we have for the proper time

$$d\tau(t) = dt - dz, \tag{68}$$

and, combining eq. (67) with eq. (68), it is

$$d\tau(t) = dt \mp dt. \tag{69}$$

Thus, it results

$$\tau_1(t) = 0 \tag{70}$$

for the forward trip
and

$$\tau_2(t) = \int_0^{T_0} 2dt = 2T_0 \tag{71}$$

for the return trip. Then

$$\tau(t) = \tau_1(t) + \tau_2(t) = 2T_0. \tag{72}$$

Thus it is $\delta\tau = \delta L_0 = 0$, i.e. there is no longitudinal effect. This is a direct conseguence of the fact that a SGW propagates at the speed of light. In this way in the forward trip the photon travels at the same speed of the SGW and its proper time is equal to zero (eq. (70)), while in the return trip the photon travels **against** the SGW and its proper time redoubles (eq. (71)).

5 The detector pattern in the SNN gauge

If the arms of our interferometer are in general in the $\overrightarrow{u}$ and $\overrightarrow{v}$ directions, to compute the line element in the $\overrightarrow{u}$ and $\overrightarrow{v}$ directions, a spatial rotation of the SNN coordinates (2) has to be made:

$$\begin{array}{ccc} u & = & -x\cos\theta\cos\phi + y\sin\phi + z\sin\theta\cos\phi \\ \\ v & = & -x\cos\theta\sin\phi - y\cos\phi + z\sin\theta\sin\phi \\ \\ w & = & x\sin\theta + z\cos\theta, \end{array} \tag{73}$$

or, in terms of the x, y, z frame:

$$\begin{array}{ccc} x & = & -u\cos\theta\cos\phi - v\cos\theta\sin\phi + w\sin\theta \\ \\ y & = & u\sin\phi - v\cos\phi \\ \\ z & = & u\sin\theta\cos\phi + v\sin\theta\sin\phi + w\cos\theta. \end{array} \tag{74}$$

In this way the SGW is propagating from an arbitrary direction $\overrightarrow{r}$ to our interferometer (see figure 3).

We can write for the metric tensor (see Chap. (10) of ref. [14]):

$$g^{ik} = \frac{\partial x^i}{\partial x'^l}\frac{\partial x^k}{\partial x'^m}g'^{lm}. \tag{75}$$

Using eq. (73), eq. (74) and eq. (75), in the new rotated frame, the line element (2) in the $\overrightarrow{u}$ direction becomes (here we can neglect the v and w directions because bouncing photons will be used and the photon deflection

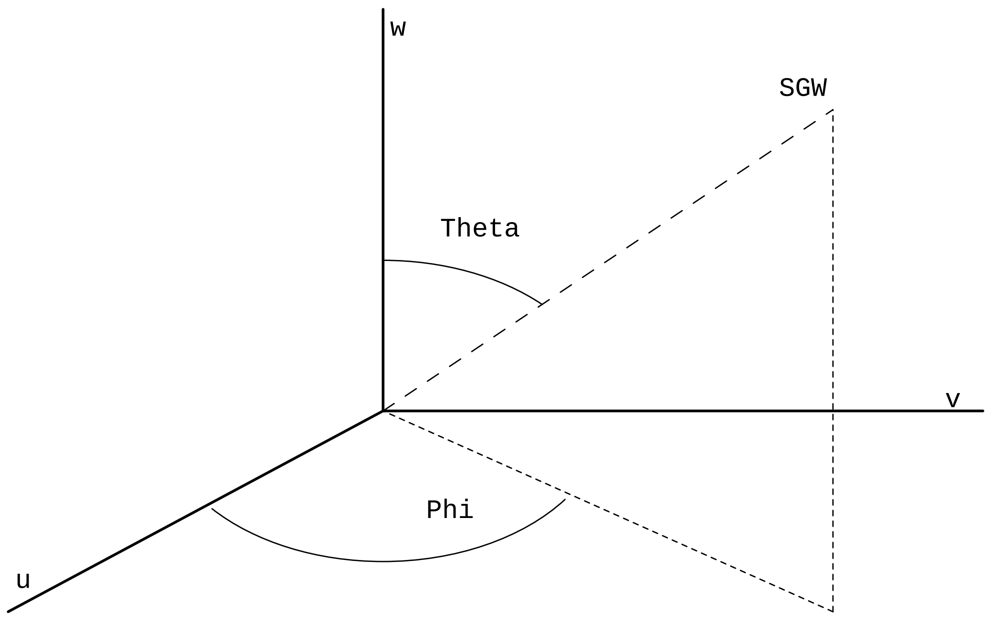

Figure 3: a SGW incoming from an arbitrary direction

into the v and w directions will be at most of order Φ, then, to first order in Φ, the dv^2 and dw^2 terms can be neglected):

$$ds^2 = [1 + (1 - \sin^2\theta\cos^2\phi)\Phi(t - u\sin\theta\cos\phi)](du^2 - dt^2). \tag{76}$$

Considering a photon launched from the beam-splitter to be bounced back by the mirror, the condition for null geodesics ($ds^2 = 0$) in eq. (76) gives

$$du^2 = dt^2. \tag{77}$$

Thus, also in this case, the analysis of [4] cannot be used starting directly from the condition of null geodesic. But a generalization of our previous analysis can be used. In fact also the metric (76) is not a “synchrony coordinate system” , thus, also in this line element, t is only a time coordinate. The rate $d\tau$ of the proper time is related to the rate dt of the time coordinate from eq. (55).

From eq. (76) it is

$$g_{00} = [1 + (1 - \sin^2\theta\cos^2\phi)\Phi(t - u\sin\theta\cos\phi)]. \tag{78}$$

.

Then, using eq. (77), it results

$$d\tau^2 = [1 + (1 - \sin^2\theta\cos^2\phi)\Phi(t - u\sin\theta\cos\phi)]du^2 \tag{79}$$

which gives

$$d\tau = \pm[1 + (1 - \sin^2\theta\cos^2\phi)\Phi(t - u\sin\theta\cos\phi)]^{\frac{1}{2}}du. \tag{80}$$

We put the beam splitter in the origin of the new coordinate system (i.e. $u_b = 0$, $v_b = 0$, $w_b = 0$). From eqs. (29) it results that an inertial test mass initially at rest in the $x - y$ plane of the SNN coordinates, remains at rest throughout the entire passage of the SGW. We also know that the coordinates of the beam-splitter and of the mirror change under the influence of the SGW in the z direction, but this fact does not influence the total variation of the round trip proper time of the photon (eq. (72)). Then, in the computation of the variation of the proper distance in the SNN gauge, the coordinates of the beam-splitter $u_b = 0$ and of the mirror $u_m = L_0$ can be considered fixed even in the $u - v$ plane, because the rotation (73) does not change the situation. Thus the proper duration of the forward trip can be found as

$$\tau_1(t) = \int_0^{L_0}[1 + (1 - \sin^2\theta\cos^2\phi)\Phi(t - u\sin\theta\cos\phi)]^{\frac{1}{2}}du \tag{81}$$

with

$$t' = t - (L_0 - u).$$

In the last equation t' is the retardation time (see Section 4).
To first order in Φ this integral can be approximated with

$$\tau_1(t) = T_0 + \frac{1 - \sin^2\theta\cos^2\phi}{2}\int_0^{L_0}\Phi(t' - u\sin\theta\cos\phi)du, \tag{82}$$

where

$$T_0 = L_0$$

is the transit time of the photon in the absence of the SGW. Similiary, the duration of the return trip will be

$$\tau_2(t) = T_0 + \frac{1 - \sin^2\theta\cos^2\phi}{2}\int_{L_0}^{0}\Phi(t' - u\sin\theta\cos\phi)(-du), \tag{83}$$

though now the retardation time is

$$t' = t - (u - l).$$

The round-trip time will be the sum of $\tau_2(t)$ and $\tau_1[t - T_0]$. Thus, to first order in Φ, the proper duration of the round-trip will be

$$\tau_{r.t.}(t) = \tau_1[t - T_0] + \tau_2(t). \tag{84}$$

Using eqs. (82) and (83) it appears immediately that deviations of this round-trip time (i.e. proper distance) from its imperurbated value are given by

$$\delta\tau(t)=\frac{1-\sin^2\theta\cos^2\phi}{2}\int_0^{L_0}[\Phi(t-2L_0+u(1-\sin\theta\cos\phi))+ \\ +\Phi(t-u(1+\sin\theta\cos\phi))]du. \tag{85}$$

Now, using the Fourier transform of our scalar field defined by eq. (64), in the frequency domain it is:

$$\delta\tilde{\tau}(\omega)=(1-\sin^2\theta\cos^2\phi)\tilde{H}_u(\omega,\theta,\phi)\Phi(\omega) \tag{86}$$

where

$$\tilde{H}_u(\omega,\theta,\phi)=\frac{-1+\exp(2i\omega L_0)}{2i\omega(1-\sin^2\theta\cos^2\phi)}+ \\ +\frac{\sin\theta\cos\phi((1+\exp(2i\omega L_0)-2\exp i\omega L_0(1+\sin\theta\cos\phi)))}{2i\omega(1-\sin^2\theta\cos^2\phi)}, \tag{87}$$

and it immediately results that $\tilde{H}_u(\omega,\theta,\phi)\to L_0$ when $\omega\to 0$.

Thus the total response function of the arm of the interferometer in the $\overrightarrow{u}$ direction to the SGW is:

$$\Upsilon_u^{SGW}(\omega)=\frac{1-\sin^2\theta\cos^2\phi}{L_0}\tilde{H}_u(\omega,\theta,\phi). \tag{88}$$

In the same way the line element (2) in the $\overrightarrow{v}$ direction becomes:

$$ds^2=[1+(1-\sin^2\theta\sin^2\phi-)\Phi(t-v\sin\theta\sin\phi)](dv^2-dt^2), \tag{89}$$

and, with the same type of analysis used for the $\overrightarrow{u}$ direction, the response function of the $\overrightarrow{v}$ arm of the interferometer to the SGW results:

$$\Upsilon_v^{SGW}=\frac{1-\sin^2\theta\sin^2\phi}{L_0}\tilde{H}_v(\omega,\theta,\phi), \tag{90}$$

where

$$\tilde{H}_v(\omega,\theta,\phi)=\frac{-1+\exp(2i\omega L_0)}{2i\omega(1-\sin^2\theta\sin^2\phi)}+ \\ +\frac{\sin\theta\sin\phi((1+\exp(2i\omega L_0)-2\exp i\omega L(1+\sin\theta\sin\phi)))}{2i\omega(1-\sin^2\theta\sin^2\phi)}, \tag{91}$$

and we see that also $\tilde{H}_v(\omega,\theta,\phi)\to L_0$ when $\omega\to 0$.

Then the detector pattern of an interferometer to the SGW is:

$$\tilde{H}^{SGW}(\omega)=\frac{1-\sin^2\theta\cos^2\phi}{L_0}\tilde{H}_u(\omega,\theta,\phi)-\frac{1-\sin^2\theta\sin^2\phi}{L_0}\tilde{H}_v(\omega,\theta,\phi)= \\ =\frac{\sin\theta}{2i\omega L}\{\cos\phi[1+\exp(2i\omega L)-2\exp i\omega L(1+\sin\theta\cos\phi)]+ \\ -\sin\phi[1+\exp(2i\omega L_0)-2\exp i\omega L_0(\sin\theta\sin\phi-1)]\}, \tag{92}$$

that in the low frequencies limit ($\omega \to 0$) is in perfect agreement with the detector pattern of eq. (15) of [2], where the computation was made in the TT gauge, and also with the low frequencies detector pattern of [15]:

$$\tilde{H}^{SGW}(\omega \to 0) = -\sin^2\theta\cos 2\phi. \tag{93}$$

Thus, it has been shown that a generalization of the analysis of [4] works in the computation of the detector pattern of interferometers for a SGW.

The detector pattern of eq. (92) is different from the one in eq. (5.25) of [3], because in [3] the computation was made starting from wrong equations of motion. The similiarity between the two detector patterns is due to the fact that the correct equations of motion (29) for a progressive SGW are casually very similiar to the wrong ones (32) for a regressive SGW used in [3] (see also comments in Section 3 about the casual similiarity between wrong and correct equations of motion).

The absolute value of the total response function of the Virgo interferometer ($L = 3$ Km) for SGWs with $\theta = \frac{\pi}{4}$ and $\phi = \frac{\pi}{3}$ and the angular dependence of the response of the Virgo interferometer for a SGW with a frequency of $f = 100Hz$ are respectively shown in figs. 4 and 5.

It has to be emphasized that the angular dependence was not examined in the purely tensorial case shown in ref. [4].

6 Conclusions

While the response of interferometers to tensorial gravitational waves has been calculated in lots of works, the coupling between interferometers and scalar gravitational waves is a more recent field of interest and it has not an analogous number of works in literature.

In the first work of [1] the authors did not realize that, in their gauge, the beam splitter is not left at the origin by the passage of the SGW, and furthermore computed a coordinate-time interval than a proper-time interval, reaching the incorrect conclusion that the SGW has longitudinal effect, and does not have transverse one. The authors of [2] showed the transverse effect of SGWs in the SNN gauge of [1]. After this, in [3], the analysis of [2] was generalized with the computation of the frequency-dependent angular pattern of interferometers in the SNN gauge, while in [2] the angular pattern was only computed in the low frequencies approximation.

In the present work the SNN gauge has been reanalyzed, showing that in [1] there were errors in the geodesic equations of motion too. These errors reflected also in [2] and [3] where wrong equation of motion taken from [1] were used.

In the analysis of the coupling between interferometers and SGWs the computation has first been made in the low frequencies approximation exactly like in [2], then the analysis has been generalized to all SGWs. In the generalization to all frequencies, with a transform of the time coordinate to the proper time, a generalization to the SNN gauge of the analysis of [4] has been used (in [4] the analysis was made for tensorial waves in the TT coordinate system).

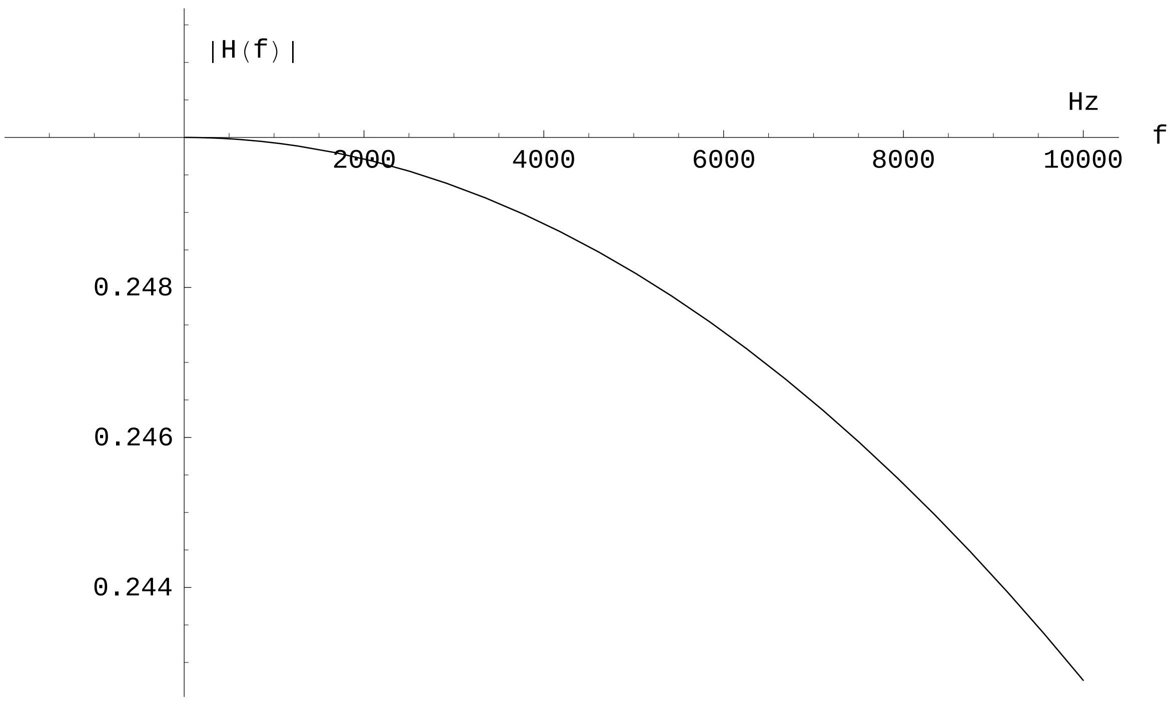

Figure 4: the absolute value of the total response function of the Virgo interferometer to SGWs for $\theta = \frac{\pi}{4}$ and $\phi = \frac{\pi}{3}$.

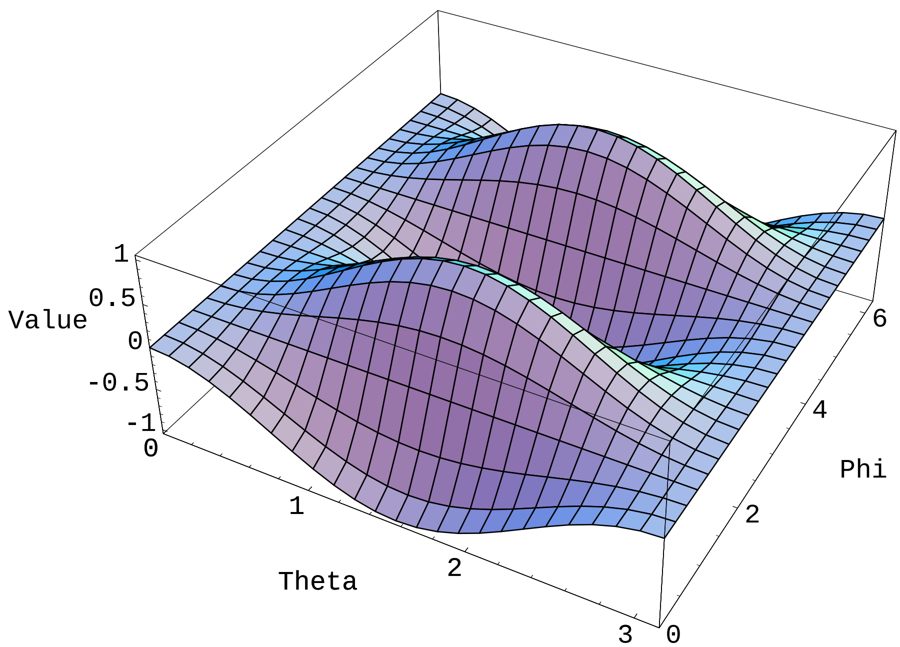

Figure 5: the angular dependence of the response of the Virgo interferometer for a SGW with a frequency of $f = 100Hz$

At the end of this paper, the correct detector pattern of interferometers in the SNN gauge has been computed with a further generalization of the analysis of [4] to the angular dependence of the propagating SGW. Our result is in contrast with the detector pattern of [3]: the similiarity between the two detector patterns is due to the fact that the correct equations of motion for a progressive SGW given here are casually very similiar to the wrong ones for a regressive SGW used in [3]. In the long wavelenghts limit our detector pattern is in perfect agreement with the detector pattern of [2], where the computation was made in the TT gauge, and of [15].

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References

[1] Shibata M, Nakao K and Nakamura T - Phys. Rev. D **50**, 7304 (1994)

[2] Maggiore M and Nicolis A - Phys. Rev. D **62** 024004 (2000); also in gr-qc/9907055

[3] Nakao K, Harada T, Shibata M, Kawamura S and Nakamura T - Phys. Rev. D 63, 082001 (2001)

[4] Rakhmanov M - Phys. Rev. D **71** 084003 (2005)

[5] http://www.ligo.org/pdf_public/camp.pdf

[6] http://www.ligo.org/pdf_public/hough02.pdf

[7] Capozziello S - *Newtonian Limit of Extended Theories of Gravity* in *Quantum Gravity Research Trends* Ed. A. Reimer, pp. 227-276 Nova Science Publishers Inc., NY (2005) arXiv:gr-qc/0412088

[8] Capozziello S and Troisi A Phys. Rev. D **72** 044022 (2005)

[9] Brunetti M, Coccia E, Favone V and Fucito F - Phys. Rew. D **59** 044027 (1999)

[10] Gasperini M and Veneziano G - Phys. Rev. D **50** 2519 (1994)

[11] Gasperini M and Ungarelli C Phys. rev. D **64** 064009 (2001)

[12] Misner CW, Thorne KS and Wheeler JA - “Gravitation” - W.H.Feeman and Company - 1973

[13] Maggiore M- Physics Reports **331**, 283-367 (2000)

[14] Landau L and Lifsits E - “Teoria dei campi” - Editori riuniti edition III (1999)

[15] Bonasia N and Gasperini G Phys. Rew. D **71** 104020 (2005)